# ON THE DISTANCE COEFFICIENT BETWEEN ISOMORPHIC FUNCTION SPACES*

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ABSTRACT

If $X$, $Y$ are compact countable metric spaces such that $Y$ contains no subset homeomorphic to $X$, then for any isomorphism $\phi$ of $C(X)$ into $C(Y)$, $\| \phi \| \| \phi^{-1} \| \geqq 3$. This result and some variants of it are established here, and prove a special case of a conjecture raised in [1].

## 1. Introduction

If $X$ is a locally compact Hausdorff space, $C_0(X)$ will denote the Banach space (with the usual sup. norm) of continuous real valued functions defined on $X$ which vanish at infinity, that is, for every $\varepsilon > 0$ and $f \in C_0(X)$ $\{x \in X; |f(x)| \geqq \varepsilon\}$ is a compact set. If $X$ is compact we write $C(X)$ instead of $C_0(X)$.

D. Amir [1] proved the following generalization of the Banach Stone theorem:

(1.1) *If $X, Y$ are non-homoeomorphic compact Hausdorff spaces, and $\phi$ is any isomorphism of $C(X)$ onto $C(Y)$, then $\| \phi \| \| \phi^{-1} \| \geqq 2$.*

Since there are no known examples of non-homeomorphic compact $X, Y$ which admit "onto" isomorphisms with $2 \leqq \| \phi \| \| \phi^{-1} \| < 3$, D. Amir conjectured that the number 2 may be replaced by 3 in this theorem.

Using a different method of proof M. Cambern [3] showed that:

(1.2) *If $X, Y$ are non-homeomorphic locally compact Hausdorff spaces, and if $\phi$ is an isomorphism of $C_0(X)$ onto $C_0(Y)$, then $\| \phi \| \| \phi^{-1}\| \geqq 2$.*

Here the number 2 could not be improved upon, for Cambern [5] constructed an example in which 2 was exact.

In the sequel we shall adopt the following notations: Given a locally compact

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space $X$, $\varepsilon > 0$ and $f \in C_0(X)$, $K(f,\varepsilon)$ will denote the compact set $\{x \in X; |f(x)| \geqq \varepsilon\}$. If $S$ is some subset of $X$, $f/S$ will denote the restriction of $f$ to $S$.

For any ordinal number $\alpha$, the $\alpha$th derivative of $X$, $X^{(\alpha)}$ is defined by transfinite induction: $X^{(0)} = X$, $X^{(1)}$ is the set of non-isolated points of $X$, and

$$X^{(\alpha)} = \begin{cases} (X^{(\beta)})^{(1)}; & \alpha = \beta + 1 \\ \bigcap_{\beta<\alpha} X^{(\beta)}; & \text{otherwise.} \end{cases}$$

If $\alpha,\beta$ are ordinals $[\alpha,\beta]$ (respectively $[\alpha,\beta)$) will denote the set of all ordinals such that $\alpha \leqq \lambda \leqq \beta$ (respectively $\alpha \leqq \lambda < \beta$). $|S|$ will denote the cardinality of a set $S$, and $\bar{S}$ its closure (if $S$ is a subset of a topological space). The empty set is denoted by $\Phi$.

§2 is devoted to the proof of our main theorem The idea of its proof originates from [4]. In §3 we bring some applications and examples. All the results here are easily seen to apply for complex function spaces as well.

## 2. The Main Theorem

(2.1) THEOREM. *Let $X$ and $Y$ be locally compact Hausdorff spaces and let $\phi: C_0(X) \xrightarrow{\text{into}} C_0(Y)$ be an isomorphism. If there is an ordinal $\alpha$ such that $|X^{(\alpha)}| > |Y^{(\alpha)}|$, then $\|\phi\| \, \|\phi^{-1}\| \geqq 3$.*

For the proof of Theorem (2.1) we need the following lemma:

(2.2) LEMMA. *Let $X$ and $Y$ be locally compact Hausdorff spaces, and let $\phi: C_0(X) \xrightarrow{\text{into}} C_0(Y)$ be a norm increasing linear map such that $\|\phi\| < 3$. Set $\varepsilon = (3(1-\eta) - \|\phi\|)/2$, where $0 < \eta < 1$ and $3(1-\eta) > \|\phi\|$. Then:*

(2.2.1) *If $f, g, h \in C_0(X)$ satisfy the conditions:*

(i) $|h| \leqq |f| \leqq |g| \leqq 1$,

(ii) $gh \geqq 0$, $gf \geqq 0$,

(iii) $\|h\| > 1-\eta$.

*Then, $K(\phi f, \varepsilon) \supseteq K(\phi(g+2h), 3(1-\eta)) \neq \Phi$.*

(2.2.2) *If for $g, h \in C_0(X)$ and some ordinal $\beta$ we have that: $|h| \leqq |g| \leqq 1$, $hg \geqq 0$ and $\|h/X^{(\beta)}\| > 1-\eta$. Then:*

(*) $\bigcap_f K(\phi f, \varepsilon) \cap Y^{(\beta)} \neq \Phi$, *where the intersection is taken over all $f \in C_0(X)$ such that $|h| \leqq |f| \leqq |g|$ and $fg \geqq 0$.*

(2.2.3) *If $f \in C_0(X)$ and for some ordinal $\beta$, $\|f/X^{(\beta)}\| = \|f\|$ then $\|\phi f/Y^{(\beta)}\| \geqq \varepsilon \|f\|$.*

PROOF. (2.2.1): If $f, g$ $h \in C_0(X)$ satisfy conditions (i)–(iii), then obviously

$\| g + 2h \| > 3(1-\eta)$, hence $\| \phi(g+2h) \| \geqq \| g+2h \| > 3(1-\eta)$, so that $K(\phi(g+2h),\ 3(1-\eta))$ is not empty.

Observe next that $\|g + 2h - 2f\| \leqq 1$. We have therefore, if

$$y_0 \in K(\phi(g+2h),\ 3(1-\eta))$$

$$\| \phi \| \geqq \| \phi(g+2h-2f) \| \geqq |\phi(g+2h)(y_0) - 2(\phi f)(y_0)|$$
$$\geqq |\phi(g+2h)(y_0)| - 2|\phi f(y_0)| \geqq 3(1-\eta) - 2|\phi f(y_0)|,$$

hence,

$$|\phi f(y_0)| \geqq 3(1-\eta)/2 - \| \phi \|/2 = \varepsilon.$$

(2.2.2): We prove this by transfinite induction on $\beta$. For $\beta = 0$, this is a consequence of (2.2.1). Suppose (2.2.2) is true for all $\delta < \beta$, and we prove it for $\beta$. There are two cases: (i) $\beta$ is a limit ordinal, (ii) $\beta = \gamma + 1$.

In (i), by the induction hypothesis $\bigcap_f K(\phi f, \varepsilon) \cap Y^{(\delta)} \neq \Phi$ for all $\delta < \beta$, hence by compactness $\bigcap_f K(\phi f, \varepsilon) \cap Y^{(\beta)} = \bigcap_f K(\phi f, \varepsilon) \cap \bigcap_{\delta<\beta} Y^{(\delta)} \neq \Phi$.

In (ii), in order to prove (*) it is sufficient to show that $\bigcap_f K(\phi f, \varepsilon) \cap Y^{(\gamma)}$ is an infinite set. Let $x \in X^{(\beta)}$ be such that $|h(x)| > 1-\eta$. There is then an infinite set $\{x_i\}_{i=1}^{\infty}$ of distinct relatively isolated points of $X^{(\gamma)}$, such that $|h(x_i)| > 1-\eta$ for all $i$. By the Urysohn lemma we can construct a set $\{h_i;\ i = 1,2,\cdots\} \subset C_0(X)$ such that $0 \leqq h_i \leqq 1$, $h_i(x_i) = 1$ and $h_i h_j = 0$ for $i \neq j$.

Denote,

$$A = \cap \ \{K(\phi f, \varepsilon);\ f \in C_0(X),\ |h| \leqq |f| \leqq |g|,\ fg \geqq 0\},$$
$$A_i = \cap \ \{K(\phi f, \varepsilon);\ f \in C_0(X),\ |h_i h| \leqq |f| \leqq |g|,\ fg \geqq 0\},$$

$i = 1,2,\cdots$. We have for all $i$,

(1) $A_i \subset A$ (obvious).

(2) $A_i \cap Y^{(\gamma)} \neq \Phi$ (by the induction hypothesis).

(3) The intersection of any $n$ $(> \| \phi \| / \varepsilon)$ sets of $\{A_i;\ i = 1,2,\cdots\}$ is empty.

For if $y \in \bigcap_{j=1}^{n} A_{i_j}$, where $\{i_1, i_2, \cdots, i_n\}$ is some set of distinct integers, then letting $H = \sum_{j=1}^{n} [\mathrm{sign}(\phi(h_{i_j}h)(y)] h_{i_j} h$, we have $\| H \| \leqq 1$, but

$$\| \phi \| \geqq (\phi H)(y) = \sum_{j=1}^{n} |\phi(h_{i_j}h)(y)| \geqq n\varepsilon > \| \phi \|,$$

which is a contradiction.

From (1)–(3) it follows immediately that $A \cap Y^{(\gamma)}$ is an infinite set, and the proof is concluded.

(2.2.3): This is a direct application of (2.2.2).

PROOF OF THEOREM (2.1). Assume first that $Y^{(\alpha)}$ is a finite set of $m$ points. If there is an "into" isomorphism $\phi$ such that $\|\phi\|\,\|\phi^{-1}\| < 3$, without loss of generality we may suppose that $\|\phi^{-1}\| = 1$ and then let $\varepsilon, \eta$ be as in Lemma (2.2). Choose any fixed subset $X_0 = \{x_1, x_2, \cdots, x_n\} \subseteq X^{(\alpha)}$ where $n > m$. Construct $h_j \in C_0(X)$ $j = 1,2,\cdots,n$, such that $0 \leqq h_j \leqq 1$, $h_j(x_j) = 1$ and $h_i h_j = 0$ if $i \neq j$. Define the operator $L: C(X_0) \to C_0(X)$ by

$$(Lf)(x) = \sum_{j=1}^{n} h_j(x) f(x_j) \qquad (x \in X,\ f \in C(X_0)).$$

Clearly $\|Lf\| = \|f\|$, so that if $R: C_0(Y) \to C(Y^{(\alpha)})$ is the natural restriction operator, (2.2.3) implies that $\|R\phi Lf\| \geqq \varepsilon \|f\|$ if $f \in C(X_0)$. Therefore, $R\phi L$ is an isomorphism mapping the $n$-dimensional space $C(X_0)$ into the $m$-dimensional space $C(Y^{(\alpha)})$, which is impossible since $n > m$.

If $Y^{(\alpha)}$ is an infinite set, let $Z = Y \cup \{\infty\}$ be the one point compactification of $Y$. $C_0(Y)$ is equivalent to the subspace of all functions of $C(Z)$ which vanish at $\infty$. Obviously, $Y^{(\alpha)} \subseteq Z^{(\alpha)} \subseteq Y^{(\alpha)} \cup \{\infty\}$, so that $|Z^{(\alpha)}| = |Y^{(\alpha)}| < |X^{(\alpha)}|$.

If $\phi: C_0(X) \xrightarrow{\text{into}} C_0(Y)$ is such that $\|\phi\|\,\|\phi^{-1}\| < 3$, again suppose that $\|\phi^{-1}\| = 1$ and let $\varepsilon, \eta$ be as in Lemma (2.2). Consider $\phi$ to be an isomorphism of $C_0(X)$ into $C(Z)$, and define the map $\sigma: X \to 2^Z$ by $\sigma(x) = \cap\,\{K(\phi f\ \varepsilon); f \in F(x)\}$, where $F(x)$ contains all the functions $f \in C_0(X)$ for which $0 \leqq f \leqq 1$, and $f(x) = 1$ (a similar map $\sigma$ was introduced in [1]). We see first that

(2.1.1) If $n > \|\phi\|/\varepsilon$, and $x_1, x_2, \cdots, x_n$ are distinct points of $X$, then $\bigcap_{i=1}^{n} \sigma(x_i) = \Phi$.

For if $y \in \bigcap_{i=1}^{n} \sigma(x_i)$, let $\{h_i;\ i = 1,2,\cdots,n\} \subset C_0(X)$ be such that $h_i \in F(x_i)$, $h_i h_j = 0$ if $i \neq j$. Then, since $|\phi h_i(y)| \geqq \varepsilon$, upon letting $H = \sum_{i=1}^{n} (\operatorname{sign}(\phi h_i)(y)) h_i$, we obtain that $\|H\| = 1$, hence

$$\|\phi\| \geqq (\phi H)(y) = \sum_{i=1}^{n} |\phi h_i(y)| \geqq n\varepsilon > \|\phi\|,$$

which is a contradiction.

(2.1.2) If $x \in X^{(\alpha)}$, then $\sigma(x) \cap Z^{(\alpha)} \neq \Phi$.

For suppose that $\sigma(x_0) \cap Z^{(\alpha)} = \Phi$ for some $x_0 \in X^{(\alpha)}$. There is, due to the compactness of $Z$, a finite empty intersection: $Z^{(\alpha)} \bigcap_{i=1}^{n} K(\phi f_i, \varepsilon) = \Phi$, with $f_i \in F(x_0)$. Put $g(t) = \max\{f_i(t); i = 1,2,\cdots,n\}$, and $h(t) = \min\{f_i(t); i = 1,2,\cdots,n\}$. Since $0 \leqq h \leqq f_i \leqq g \leqq 1$ and $h(x_0) = 1$, it follows from Lemma (2.2) that $\bigcap_{i=1}^{n} K(\phi f_i, \varepsilon) \cap Y^{(\alpha)} \neq \Phi$, which is a contradiction.

Now, on applying (2.1.1) and (2.1.2) the set $\bigcup_{x\in X^{(\alpha)}}[\sigma(x)\cap Z^{(\alpha)}]$ should contain at least $|X^{(\alpha)}|$ elements, but being a subset of $Z^{(\alpha)}$, this is impossible since $|Z^{(\alpha)}|<|X^{(\alpha)}|$.

REMARK. M. Cambern [5] constructed a simple example where $|X^{(1)}| = |Y^{(1)}| = 1$, $X$ is compact, $Y$ locally compact, and an isomorphism $\phi: C(X) \xrightarrow{\text{onto}} C_0(Y)$ such that $\|\phi\|\,\|\phi^{-1}\| = 2$. Thus we cannot replace in Theorem (2.1) the assumption "$|X^{(\alpha)}| > |Y^{(\alpha)}|$" by the weaker assumption "$X$ not homeomorphic to $Y$", even when $\phi$ is an onto isomorphism.

3. **Applications and Examples.** As first application we prove

(3.1) THEOREM. *Let $X, Y$ be compact countable metric spaces such that $Y$ contains no subset homeomorphic to $X$. Then for any isomorphism $\phi: C(X) \to C(Y)$, $\|\phi\|\,\|\phi^{-1}\| \geqq 3$.*

PROOF: According to [6], every compact countable metric space $Z$ is homeomorphic to some interval of ordinals $[1,\omega^{\alpha}.n]$ with the order topology, where $\omega$ is the first infinite ordinal, $1 \leqq \alpha < \omega_1$, where $\omega_1$ is the first uncountable ordinal, and $1 \leqq n < \omega$. Conversely, if $1 \leqq n < \omega$ and $1 \leqq \alpha < \omega_1$, $[1,\omega^{\alpha}.n]$ is a countable compact metric space.

Suppose now that $X = [1,\omega^{\alpha}\cdot n]$ and $Y = [1,\omega^{\beta}.m]$. The condition imposed on $X$ means that either $\alpha > \beta$, or $\alpha=\beta$ and $n > m$. In either case, $Y^{(\beta)}$, which consists of exactly $m$ points, has a smaller cardinality than $X^{(\beta)}$, and the result follows from Theorem (2.1).

(3.1.1.) EXAMPLE. The number 3 is attained e.g. when $X=[1,\omega.2]$, $Y=[1,\omega]$ and $\phi: C(X) \xrightarrow{\text{onto}} C(Y)$ is defined by:

$$(\phi f)(1) = f(\omega) - f(\omega.2)$$

$$(\phi f)(2m) = f(m) - \tfrac{1}{2}[f(\omega) - f(\omega.2)], \qquad \omega > m \geqq 1$$

$$(\phi f)(2m+1) = f(\omega+m) + \tfrac{1}{2}[f(\omega) - f(\omega.2)], \quad \omega > m \geqq 1$$

$$(\phi f)(\omega) = \tfrac{1}{2}f(\omega) + \tfrac{1}{2}f(\omega.2).$$

It is easily verified that $\|\phi\| = 2$, $\|\phi^{-1}\| = 3/2$.

(3.1.2) EXAMPLE. For every isomorphism $\phi$ mapping $c(=C([1,\omega]))$ into $c_0\ (= C_0([1,\omega)))$, $\|\phi\|\,\|\phi^{-1}\| \geqq 3$. This result which is a simple consequence of Theorem (2.1) was obtained for onto isomorphisms in [4]. Here the number 3 is again exact, and is obtained for the isomorphism $\phi$ defined by:

$$(\phi f)(1) = 2f(\omega)$$

$$(\phi f)(n+1) = f(n) - f(\omega), \qquad \omega > n \geqq 1.$$

Theorem (3.1) is obviously not true when $X$ and $Y$ are uncountable compact metric spaces, for it is then well known that $C(Y)$ is a universal Banach space i.e. every separable Banach space is isometrically embeddable in $C(Y)$. However, it may well be then that for "onto" isomorphisms, $\| \phi \| \, \| \phi^{-1} \| \geqq 3$ always. This is a special case of the Amir's conjecture, and its special interest lies in the fact that $C(X)$ and $C(Y)$ are isomorphic whenever $X$ and $Y$ are uncountable metric spaces ([7], [8]). Complete characterization of the isomorphic types of $C([1, \omega^\alpha . n])$ spaces, $1 \leqq \alpha < \omega_1$, $1 \leqq n < \omega$, is given in [2].

Recall that a subset $S \neq \Phi$ of a topological space $X$ is called perfect if $S^{(1)} = S$. It is obvious and well known that $PX = \bigcap_{\alpha \geqq 1} X^{(\alpha)}$ is the maximal perfect subset of a space $X$, and is called the perfect kernel of $X$.

(3.2) LEMMA. *Let $X$ and $Y$ be locally compact Hausdorff spaces and let $\phi: C_0(X) \xrightarrow{\text{onto}} C_0(Y)$ be an isomorphism such that $\| \phi \| \, \| \phi^{-1} \| < 3$. Then there is an isomorphism $\tilde{\phi}: C_0(Y \sim PY) \xrightarrow{\text{into}} C_0(\overline{X \sim PX})$ such that $\| \tilde{\phi} \| \, \| \tilde{\phi}^{-1} \| \leqq \| \phi \| \, \| \phi^{-1} \|$.*

PROOF. Let $L: C_0(Y \sim PY) \to C_0(Y)$ be the isometry (into) defined by

$$(Lf)(y) = \begin{cases} f(y); & y \in Y \sim PY \\ 0; & y \in PY \end{cases}$$

($f \in C_0(Y \sim PY)$). Without loss of generality assume that $\| \phi^{-1} \| = 1$.

If $0 \neq f \in C_0(Y \sim PY)$, let $g = \phi^{-1} Lf / \| \phi^{-1} Lf \|$. We have that $\| g \| = 1$, but $\| g / PX \| < 1$ (otherwise by (2.2.3) $\| \phi g / Y^{(\alpha)} \| \geqq \varepsilon$ for every ordinal $\alpha$, that is $K(\phi g, \varepsilon) \cap Y^{(\alpha)} \neq \Phi$, and by compactness $K(\phi g, \varepsilon) \cap PY \neq \Phi$, which is clearly impossible). Therefore if $R: C_0(X) \to C_0(\overline{X \sim PX})$ is the natural restriction operator, then $\| Rg \| = 1$, that is $\| R\phi^{-1} Lf \| = \| \phi^{-1} Lf \|$, but

$$\| \phi \|^{-1} \| f \| \leqq \| \phi^{-1} Lf \| \leqq \| f \|,$$

whence letting $\tilde{\phi} = R\phi^{-1}L$, the statement in the lemma now becomes obvious.

(3.3) COROLLARY. *Let $X, Y, \phi$ be as in Lemma (3.2). Then for every ordinal $\alpha$,*

$$| (Y \sim PY)^{(\alpha)} | \leqq | (\overline{X \sim PX})^{(\alpha)} |.$$

PROOF. Consider the isomorphism $\tilde{\phi}$ of Lemma (3.2) and apply Theorem (2.1).

It is easy to construct examples of spaces with non empty perfect kernels where 3 is exact, e.g.

(3.4) EXAMPLE. $X = [0,1]$, $Y = [0,1) \cup \{2\}$. Define $\phi: C(X) \xrightarrow{\text{onto}} C_0(Y)$ by

$$(\phi f)(2) = 2f(1)$$

$$(\phi f)(x) = f(x) - f(1), \qquad 0 \leqq x < 1.$$

Clearly $\| \phi \| = 2$, $\| \phi^{-1} \| = \frac{3}{2}$. Also, it follows from Corollary (3.3) that for every isomorphism $\psi$ of $C(X)$ onto $C_0(Y)$, $\| \psi \| \, \| \psi^{-1} \| \geqq 3$.

## REFERENCES


1. D. Amir, *On isomorphisms of continuous function spaces*, Israel J. Math. **3** (1965), 205–210.
2. C. Bessaga and A. Pełczyński, *Spaces of continuous functions (IV)*, Studia Math. **19** (1960), 53–62.
3. M. Cambern, *On isomorphisms with small bound*, Proc. Amer. Math. Soc. **18** (1967), 1062–1066.
4. M. Cambern, *On mappings of sequence spaces*, Studia Math **30** (1968), 73–77.
5. M. Cambern, *Mappings of continuous function spaces*, Notices Amer. Math. Soc. **16** (1969), 317.
6. S. Mazurkiewicz and W. Sierpinski, *Contributions à la topologie des ensembles denombrables*, Fund. Math, **1** (1920), 17–27.
7. A. A. Miljutin, *Isomorphism of spaces of continuous functions on compacts of the power continuum*, Teor. Funk. Funkcional. Analiz i Priložen. **2** (1966), 150–156 (Russian).
8. A. Pełczyński, *Linear extensions, linear averagings and their application to linear topological classification of spaces of continuous functions*, RozpraWy Matematyczne **58**, (1968).


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